International Journal of Theoretical Physics, Vol. 44, No. 9, September 2005 (© 2005)
DOI: 10.1007/s10773-005-4779-6

Measures Defined on Quantum Logics of Sets

A. Aizpuru,[1] A. Gutiérrez-Dávila,[1] and Wu Junde[2,3]

Received January 11, 2005; accepted March 30, 2005

We study families formed with subsets of any set X which are quantum logics but which are not Boolean algebras. We consider sequences of measures defined on a sets quantum logics and valued on an effect algebra and obtain a sufficient condition for a sequences of such measures to be uniformly strongly additive with respect to order topology of effect algebras.

KEY WORDS: quantum algebras; measures; natural families.

1. INTRODUCTION

A structure $(L, \oplus, 0, 1)$ is called an *effect algebra* if 0, 1 are two distinguished elements and $\oplus$ is a partially defined operation on L which satisfies the following conditions for any $a, b, c \in L$ (Foulis and Bennett, 1994):

(1) $b \oplus a = a \oplus b$ if $a \oplus b$ is defined (it is said that a and b are orthogonal elements).
(2) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ if one side is defined.
(3) For every $a \in L$ there exists a unique $b \in L$ such that $a \oplus b = 1$ (we denote b by a').
(4) If $1 \oplus a$ is defined then $a = 0$.

In effect algebra L we consider the following partial order: $a \leq b$ iff there exists $c \in L$ such that $a \oplus c = b$ (write $c = b - a$).

If for all $a, b \in L$, $a \leq b$ or $b \leq a$, then L is said to be *totally order effect algebra*. If for all $a, b \in L$ and $a < b$ (which means $a \leq b$ and $a \neq b$) there exists $c \in L$ such that $a < c < b$, then L is said to be connected.

[1] Departamento de Matemáticas, Universidad de Cádiz, Apdo–40, 11510-Puerto Real (Cádiz), Spain; e-mail: antonio.aizpuru@uca.es.
[2] Department of Mathematics, Zhejiang University, Hangzhou 310027, People's Republic of China.
[3] To whom correspondence should be addressed at Department of Mathematics, Zhejiang University, Hangzhou 310027, People's Republic of China; e-mail: wjd@math.zju.edu.cn.

Let $F = \{a_i : 1 \leq i \leq n\}$ be a finite subset of L. If $a_1 \oplus a_2$, $(a_1 \oplus a_2) \oplus a_3, \ldots (a_1 \oplus a_2 \oplus \ldots \oplus a_{n-1}) \oplus a_n$ are defined, we say that F is *orthogonal* and we denote $\bigoplus F = (a_1 \oplus a_2 \oplus \ldots \oplus a_{n-1}) \oplus a_n$.

If G is an arbitrary subset of L, we will say that G is *orthogonal* if each finite subset $F \subseteq G$ is orthogonal.

If G is orthogonal and the supremum $\bigvee\{\bigoplus F : F \subseteq G,\ F \text{ finite}\}$ exists, then $\bigoplus G = \bigvee\{\bigoplus F : F \subset G,\ F \text{ finite}\}$ is called the $\oplus$-*sum* of G.

L is said to be *complete* if $\bigoplus G$ exists for each orthogonal subset $G \subseteq L$.

L is σ-*complete* if $\bigoplus G$ exists for each countable orthogonal subset $G \subseteq L$.

For the elementary properties of the order topology of effect algebra $(L\oplus, 0, 1)$, (see, Birkhoff, 1948; Riecanova, 2000; Wu *et al.*, 2005).

In this paper we will need the following results in relation to an effect algebra $(L, \oplus, 0, 1)$:

(i) If $c \leq b$ and $b \leq a$ then $c \leq a$ and $(a \ominus c) \ominus (b \ominus c) = a \ominus b$ (Foulis and Bennett, 1994).

(ii) If $a, b \in L$, let $a - b$ denote the element $a \ominus b$ if $a \geq b$ and the element $b \ominus a$ if $a \leq b$. If $\{a_1, a_2, \ldots, a_n\}$ and $\{b_1, b_2, \ldots, b_n\}$ are two orthogonal subsets of L and L is a totally order effect algebra, then

$$\bigoplus_{i=1}^{n} a_i - \bigoplus_{i=1}^{n} b_i = \bigoplus_{i\in A^+}(a_i - b_i) - \bigoplus_{i\in A^-}(b_i - a_i),$$

with $A^+ = \{i \in \mathbb{N} : b_i \leq a_i\}$ and $A^- = \{i \in \mathbb{N} : a_i < b_i\}$.

If L is also a σ-complete effect algebra and $(a_i)_i$ and $(b_i)_i$ are two orthogonal sequences, then we have

$$\bigoplus_{i=1}^{\infty} a_i - \bigoplus_{i=1}^{\infty} b_i = \bigoplus_{i\in A^+}(a_i - b_i) - \bigoplus_{i\in A^-}(b_i - a_i).$$

with $A^+ = \{i \in \mathbb{N} : b_i \leq a_i\}$ and $A^- = \{i \in \mathbb{N} : a_i < b_i\}$ (Aizpuru *et al.*, 2005).

(iii) For each $a, b, c \in L$ such that $b \ominus a = c \ominus a$ we have $b = c$. If $a \oplus b = a \oplus c$ then $b = c$ (Foulis and Bennett, 1994).

(iv) If L is a totally order effect algebra and $a, b, c \in L$, then it follows that $(a - c) \leq (a - b) \oplus (b - c)$ (Aizpuru *et al.*, 2005).

Recently, great interest has been taken in measure theory and matrix results in the framework of effect algebras (Wu *et al.*, 2003; Wu and Ma, 2003; Wu *et al.*, 2003; Aizpuru *et al.*, 2005; Mazario, 2001).

2. QUANTUM ALGEBRA OF SETS

Let X be a set and let $\mathcal{F}$ be a Boolean algebra formed with the subsets of X. If $a, b \in \mathcal{F}$ and $a \cap b = \emptyset$, define $a \oplus b = a \cup b$.

Let $\mathcal{F}$ be a family formed with subsets of X. We will say $\mathcal{F}$ is a *quantum algebra* of sets (or a quantum algebra formed with sets) if $\{\emptyset, X\} \subseteq \mathcal{F}$ and

$(\mathcal{F}, \bigoplus, X, \emptyset)$ is an effect algebra with $\bigoplus$ defined by $a \oplus b = a \cup b$ if $a \cap b = \emptyset$ (in the literature a quantum algebra is also called a class (Gudder, 1979) or a partial field (Godowski, 1981)). Let us observe that a sequence $(a_i)_i$ in a quantum algebra of sets $\mathcal{F}$ is orthogonal iff $(a_i)_i$ is a sequence of mutually disjoint subset in $\mathcal{F}$.

We next give a quantum algebra of sets which is not a Boolean algebra.

Example 2.1. Let $P \subseteq \mathbb{N}$ be the set of even numbers and let $I \subseteq \mathbb{N}$ be the set of odd numbers. Let $\mathcal{L}$ be the family of sets $A \subseteq \mathbb{N}$ such that $A \cap P$, $A \cap I$, $A^c \cap P$ and $A^c \cap I$ are infinite. Denote $\phi(\mathbb{N}) = \{A \subseteq \mathbb{N} : A \text{ is finite or cofinite}\}$. Define $(\mathcal{F}, \bigoplus, \mathbb{N}, \emptyset)$ with $\mathcal{F} = \mathcal{L} \cup \phi(\mathbb{N})$ and $a \oplus b = a \cup b$ if $a \cap b = \emptyset$, this structure is a quantum algebra and it is easily seen that $\mathcal{F}$ is not a Boolean algebra.

A set $\mathcal{F} \subseteq P(\mathbb{N})$ is called a *natural family* if $\phi_0(\mathbb{N}) \subseteq \mathcal{F}$, where $\phi_0(\mathbb{N})$ denotes the family of finite subsets of $\mathbb{N}$ (Aizpuru and Gutierrez-Davila, 2004a).

We will say $\mathcal{F}$ is a *quantum natural algebra* if $\mathcal{F}$ is a quantum algebra of sets and also a natural family.

The quantum natural algebras can be defined by any orthogonal sequence in a effect algebra:

Let $(L, \bigoplus, 1, 0)$ be an effect algebra and let $(a_n)_n$ be a orthogonal sequence in L satisfying $\oplus_n a_n$ exists. Let $L^{'} = \{\oplus_{i \in M} a_i : \oplus_{i \in M} a_i$ exists and $M \subseteq \mathbb{N}\}$ and define $(\oplus_{i \in M_1} a_i) \oplus (\oplus_{i \in M_2} a_i) = \oplus_{i \in M_1 \cup M_2} a_i$ iff $M_1 \cap M_2 = \emptyset$ and $\oplus_{i \in M_1 \cup M_2} a_i$ exists in L. Under this conditions, $(L^{'}, \bigoplus, 1, 0)$ is an effect algebra with $1 = \bigoplus_{i \in \mathbb{N}} a_i$. It is clear that $L^{'}$ can be identified with a quantum algebra. This analysis let us extend to effect algebra the results obtained in the framework of quantum natural algebras.

The following definition can be found in (Aizpuru and Gutierrez-Davila, 2004b).

Definition 2.2. A natural family $\mathcal{F}$ has property S if for every pair $[(a_i)_i, (b_i)_i]$ of disjoint sequences of mutually disjoint elements in $\phi_0(\mathbb{N})$ there exists an infinite set $M \subseteq \mathbb{N}$ and $b \in \mathcal{F}$ satisfying $a_i \subseteq b$ and $b_i \cap b = \emptyset$ for each $i \in M$.

$\mathcal{F}$ has property (SC) if for each sequences $(a_i)_i$ of mutually disjoint elements in $\mathcal{F}$ there exists an infinite set $M \subseteq \mathbb{N}$ such that $\bigcup_{i \in M} a_i \in \mathcal{F}$.

Let us observe that all these properties S and (SC) can be defined in an effect algebra.

Interesting results deals with finitely additive measures defined on an effect algebra with property (SC) have been obtained (Wu and Ma, 2003; Wu *et al.*, 2003; Mazario, 2001).

The quantum algebra of sets we have introduced in Example 2.1 has property S and lacks property SC (Aizpuru and Gutierrez-Davila, 2004b).

In this paper, we prove a condition in relation to a sequence $(\mu_i)_i$ of measures defined on a quantum natural algebra and valued in an effect algebra which implies $(\mu_i)_i$ is uniformly strongly additive.

3. SEQUENCES OF STRONGLY ADDITIVE MEASURES

In this section $\mathcal{F}$ denotes a quantum natural algebra and $(L, \bigoplus, 0, 1)$ denotes a connected, totally order effect algebra. $\mu : \mathcal{F} \to L$ is said to be a *measure* if the equality $\mu(a \cup b) = \mu(a) \oplus \mu(b)$ holds for each $a, b \in \mathcal{F}$ with $a \cap b = \emptyset$. We will say μ is σ-additive if each sequence $(a_i)_i$ of mutually disjoint elements in $\mathcal{F}$ with $\bigcup_{i \in \mathbb{N}} a_i \in \mathcal{F}$ verifies the following two properties:

(a) The sequence $(\mu(a_i))_i$ is orthogonal in L;
(b) $\mu\left(\bigcup_i a_i\right) = \bigoplus_i \mu(a_i)$.

Let us observe an element $a \in \mathcal{F}$ verifies $a = \bigcup_{i \in a}\{i\}$ and so $\mu(a) = \bigoplus_{i \in a} \mu(\{i\})$.

Let $(L, \oplus, 0, 1)$ be a totally order effect algebra. We say that the sequence $(b_n)_n$ of L is a *Cauchy sequence* if for each $h \in L, 0 < h$, there exists $n_0 \in \mathbb{N}$ such that when $n_0 \leq n, m$, then $b_n - b_m < h$. We say that the sequence $(c_n)_n$ of L is a *unconditionally Cauchy sequence* (uca) if for each $h \in L$, $0 < h$, there exists $n_0 \in \mathbb{N}$ such that when $n_0 \leq n$, for every finite subset B of $\{n+1, n+2, \ldots\}$, then $\oplus_{i \in B} c_i < h$.

Lemma 3.1. *Let $\mu : \mathcal{F} \to L$ be a σ-additive measure and let $(a_i)_i$ be a sequences of mutually disjoint elements in $\mathcal{F}$. Then $(\mu(a_i))_i$ is unconditionally Cauchy (uca).*

Proof: If not, there exists $h \in L \setminus \{0\}$ and a sequence $(C_n)_n$ in $\phi_0(\mathbb{N})$ such that $\bigoplus_{i \in C_n} \mu(a_i) > h$. With this notation, the following properties hold:

(1) Define $b_n = \bigoplus_{i \in C_n} a_i$ for each $n \in \mathbb{N}$, it is obvious that $\mu(b_n) > h$.
(2) $\bigoplus_i \mu(\{i\})$ is uca and so there exists $m \in \mathbb{N}$ which satisfies $\bigoplus_{i \in C} \mu(\{i\}) < h$ if $C \subseteq \{m+1, \ldots\}$ is a finite subset.
(3) There also exists $n \in \mathbb{N}$ such that $\inf C_n > m$.

From (1) and (3) there exists $b \in \phi_0(\mathbb{N})$ such that $b \subseteq b_n$ and $\mu(b) > h$, contrary to (2).

□

Let $(\mu_i)_i$ be a sequence of L-valued measures defined on $\mathcal{F}$. $(\mu_i)_i$ is said to be *uniformly strongly additive* in $\mathcal{F}$ if each sequence $(a_i)_i$ of mutually disjoint elements in $\mathcal{F}$ we have $\lim_j \mu_i(a_j) = 0$ with respect to the order topology of L uniformly on $i \in \mathbb{N}$.

Theorem 3.2. *Let $(\mu_i)_i$ be a sequence of σ-additive L-valued measures defined on $\mathcal{F}$ and $(\mu_i(a))_i$ a Cauchy sequence for each $a \in \mathcal{F}$. If $\mathcal{F}$ has property S, then $(\mu_i)_i$ is uniformly strongly additive.*

Proof: At first, we prove that for each sequence $(b_j)_j$ of mutually disjoint elements in $\phi_0(\mathbb{N})$, $\lim_j \mu_i(b_j) = 0$ with respect to the order topology of L uniformly for $i \in \mathbb{N}$. It is enough to prove $(\mu_i(b_j))_i$ are uniformly Cauchy for $j \in \mathbb{N}$. If not, there exists $h \in L \setminus \{0\}$ such that for each $k \in \mathbb{N}$ there exists $i > j > k$ and n_k satisfying $\mu_i(b_{n_k}) - \mu_j(b_{n_k}) > h$. From this, it is clear that for each $k, m \in \mathbb{N}$ there exists $i > j > k$ and n_k satisfying $\mu_i(b_{n_k}) - \mu_j(b_{n_k}) > h$ and $\inf b_{n_k} > m$. Now, let $k_1 = 1$, by the previous assumption there exists $i_1 > j_1 > k_1$ and n_1 such that $\mu_{i_1}(b_{n_1}) - \mu_{j_1}(b_{n_1}) > h$. For $m_1 > \sup b_{n_1}$, let $h_1 \in L \setminus \{0\}$ and $\{h_1^{'}, \ldots, h_{m_1}^{'}\} \subseteq L \setminus \{0\}$ with $h_1^{'} \oplus \ldots \oplus h_{m_1}^{'} < h_1 < h$. Since $(\mu_i(\{j\}))_i$ are Cauchy sequences for $j \in \{1, 2, \ldots, m\}$, there exists i_{01} such that $\mu_p(\{j\}) - \mu_q(\{j\}) < h_j^{'}$ for each $p, q > i_{01}$ and $j \in \{1, \ldots, m_1\}$. Let C denote any subset of $\{1, \ldots, m_1\}$ and denote $C_+ = \{j \in \{1, \ldots, m_1\} : \mu_q(\{j\}) \leq \mu_p(\{j\})\}$ and $C_- = C \setminus C_+$. It is obvious that

$$\bigoplus_{j \in C_+} (\mu_p(\{j\}) - \mu_q(\{j\})) < h_1,$$

$$\bigoplus_{j \in C_-} (\mu_q(\{j\}) - \mu_p(\{j\})) < h_1,$$

$$\bigoplus_{j \in C} \mu_p(\{j\}) - \bigoplus_{j \in C} \mu_q(\{j\}) < h_1.$$

Let $k_2 > i_{01}$, similar, there exist $i_2 > j_2 > k_2$ and n_2 such that $\mu_{i_2}(b_{n_2}) - \mu_{j_2}(b_{n_2}) > h$.

It follows from Lemma 3.1 that $\left(\mu_{i_2}(\{j\})\right)_j$ and $\left(\mu_{j_2}(\{j\})\right)_j$ are unconditionally Cauchy and so there exists $m_2 > \sup b_{n_2}$ with

$$\bigoplus_{j \in D_+} (\mu_{i_2}(\{j\}) - \mu_{j_2}(\{j\})) < h_2,$$

$$\bigoplus_{j \in D_-} (\mu_{i_2}(\{j\}) - \mu_{j_2}(\{j\})) < h_2,$$

$$\bigoplus_{j \in D} \mu_{i_2}(\{j\}) - \bigoplus_{j \in D} \mu_{i_2}(\{j\}) < h_2.$$

Where $h_2 \in L \setminus \{0\}$ with $h_1 \oplus h_2 < h$, $D_+ = \{j \in D : \mu_{j_2}(\{j\}) \leq \mu_{i_2}(\{j\})\}$, $D_- = D \setminus D_+$ and D is an arbitrary finite subset of $\{m_2 + 1, \ldots\}$.

Inductively, we obtain three strictly increasing sequences $(i_r)_r$, $(j_r)_r$, and $(m_r)_r$ with $j_1 < i_1 < j_2 < i_2 < \ldots < j_r < i_r < \ldots$ such that, for $r > 1$,

(i) $b_{n_r} \subseteq \{m_{r-1}+1, \ldots, m_r\}$ and $\mu_{i_r}(b_{n_r}) - \mu_{j_r}(b_{n_r}) > h$.
(ii) $\mu_{i_r}(c) - \mu_{j_r}(c) < h_1$ for each $c \subseteq \{1, \ldots, m_{r-1}\}$.
(iii) $\mu_{i_r}(b) - \mu_{j_r}(b) < h_2$ for each $b \subseteq \{m_r+1, \ldots\}$.

Define $k_r = \{m_{r-1}+1, \ldots, m_r\} \setminus b_{n_r}$ for each $r > 1$. Since $\mathcal{F}$ has property S, the pair $[(k_r)_{r>1}, (b_{n_r})_{r>1}]$ allow us to obtain an infinite set $M \subseteq \mathbb{N}$ and $b \in \mathcal{F}$ such that $b_{n_r} \subseteq b$ and $b \cap k_r = \emptyset$ for each $r \in M$. Note that $(\mu_i(b))_i$ is a Cauchy sequence, so there exists $i_0 \in \mathbb{N}$ satisfying $\mu_p(b) - \mu_q(b) < h_3$ for each $p, q \geq i_0$, where $h_3 \in L \setminus \{0\}$ with $h_1 + h_2 + h_3 < h$. Let r satisfy $i_r > j_r > i_0$, we have

$$b = \left(\bigoplus_{\substack{j \leq m_{r-1} \\ j \in b}} \{j\}\right) \bigoplus \left(\bigoplus_{j \in b_{n_r}} \{j\}\right) \bigoplus \left(\bigoplus_{\substack{j > m_r \\ j \in b}} \{j\}\right).$$

Denote

$$a_1 = \mu_{i_r}(b) = \bigoplus_{j \in b} \mu_{i_r}(\{j\}) \qquad a_2 = \bigoplus_{\substack{j \leq m_{r-1} \\ j \in b}} \mu_{i_r}(\{j\})$$

$$a_3 = \mu_{i_r}(b_{n_r}) \qquad a_4 = \bigoplus_{j > m_r} \mu_{i_r}(\{j\})$$

$$b_1 = \bigoplus_{j \in b} \mu_{j_r}(\{j\}) \qquad b_2 = \bigoplus_{\substack{j \leq m_{r-1} \\ j \in b}} \mu_{j_r}(\{j\})$$

$$b_3 = \mu_{j_r}(b_{n_r}) \qquad b_4 = \bigoplus_{j > m_r} \mu_{j_r}(\{j\})$$

$$\alpha = a_2 \bigoplus a_4 \qquad \beta = b_2 \bigoplus b_4$$

With this notation, we have $a_1 = \alpha \bigoplus a_3$, $b_1 = \beta \bigoplus b_3$. Since $\alpha - \beta = (a_2 \bigoplus a_4) - (b_2 \bigoplus b_4)$ (there is no loss of generality in assuming $\alpha \geq \beta$), one of the following conditions is true:

(1.1) $\alpha - \beta = (a_2 - b_2) \bigoplus (a_4 - b_4)$ if $a_2 \geq b_2$ and $a_4 \geq b_4$.
(1.2) $\alpha - \beta = (a_4 - b_4) - (b_2 - a_2)$ if $a_2 < b_2$ and $a_4 \geq b_4$.
(1.3) $\alpha - \beta = (a_2 - b_2) - (a_4 - b_4)$ if $a_2 \geq b_2$ and $a_4 < b_4$.

From all these cases we conclude $\alpha - \beta \leq h_1 + h_2$.

Since $a_1 - b_1 = (\alpha \bigoplus a_3) - (\beta \bigoplus b_3)$ (we can assume $b_1 \leq a_1$), one of the following conditions is true:

(2.1) If $\alpha \geq \beta$ and $a_3 \leq b_3$ it follows that $a_1 - b_1 = (\alpha - \beta) \bigoplus (a_3 - b_3)$ and so $a_1 - b_1 > h$, which contradicts $a_1 - b_1 = \mu_{i_r}(b) - \mu_{i_r}(b) < h_3$.

(2.2) If $a_3 \leq b_3$ and $\alpha < \beta$ it follows that $a_1 - b_1 = (a_3 - b_3) - (\alpha - \beta)$ and so $a_3 - b_3 < h_1 + h_2 + h_3$, which contradicts (i).

(2.3) If $a_3 < b_3$ and $\alpha \geq \beta$ it follows that $a_1 - b_1 = (b_3 - a_3) - (\beta - \alpha)$ and so $b_3 - a_3 < h$, which is also impossible.

Thus, we have just proved that $(\mu_i(b_j))_i$ are uniformly Cauchy on $j \in \mathbb{N}$ for each sequence $(b_j)_j$ of mutually disjoint elements in $\phi_0(\mathbb{N})$.

In order to complete the proof of Theorem 3.2, let $(b_j)_j$ be a sequence of mutually disjoint elements in $\mathcal{F}$. If $(\mu_i(b_j))_i$ are not uniformly Cauchy on $j \in \mathbb{N}$, there exists $h \in L \setminus \{0\}$ such that for each $k, m \in \mathbb{N}$ there exist $i > j > k$ and n_k satisfying $\mu_i(b_{n_k}) - \mu_j(b_{n_k}) > h$ and $\inf b_{n_k} > m$.

Let $k_1 = 1$, the above assumption allow us to consider $i_1 > j_1 > k_1$ and n_1 with $\mu_{k_1}(b_{n_1}) - \mu_{i_1}(b_{n_1}) > h$.

Let $h', h_0 \in L \setminus \{0\}$ satisfy $h' \oplus h_0 < h$. By Lemma 3.1 that there exists l_1 such that $\bigoplus_{\substack{l>l_1 \\ l\in b_{n_1}}} \mu_{i_1}(\{l\}) < h'$ and $\bigoplus_{\substack{l>l_1 \\ l\in b_{n_1}}} \mu_{j_1}(\{l\}) < h'$.

Denote $a = \mu_{i_1}(b_{n_1}) = \bigoplus_{l\in b_{n_1}} \mu_{i_1}(\{l\}) = a_1 \oplus a_2$, where $a_1 = \bigoplus_{\substack{l\leq l_1 \\ l\in b_{n_1}}} \mu_{i_1}(\{l\})$ and $a_2 = \bigoplus_{\substack{l>l_1 \\ l\in b_{n_1}}} \mu_{i_1}(\{l\})$. Similarly, let $b = \mu_{j_1}(b_{n_1}) = \bigoplus_{l\in b_{n_1}} \mu_{j_1}(\{l\}) = b_1 \oplus b_2$, where $b_1 = \bigoplus_{\substack{l\leq l_1 \\ l\in b_{n_1}}} \mu_{j_1}(\{l\})$ and $a_2 = \bigoplus_{\substack{l>l_1 \\ l\in b_{n_1}}} \mu_{j_1}(\{l\})$.

Without loss of generality we can assume $a \geq b$ and so $a - b > h$. Since $a - b = (a_1 \bigoplus a_2) - (b_1 \bigoplus b_2)$, one of the following conditions must be true:

If $a_1 \geq b_1$ and $a_2 \geq b_2$ we have $a - b = (a_1 - b_1) \bigoplus (a_2 - b_2)$. From this it follows that $a_1 - b_1 > h_0$, if not, $a - b \leq h_0 \oplus h' < h$, which contradicts the inequality $a - b > h$.

If $a_1 \geq b_1$ and $b_2 \geq a_2$ we have $a - b = (a_1 - b_1) - (b_2 - a_2)$. If $a_1 - b_1 \leq h_0$ then $a - b \leq h_0 \oplus h' < h$, which is a contradiction and so $a_1 - b_1 > h_0$.

If $a_1 \leq b_1$ and $b_2 \leq a_2$ we have $a - b = (b_1 - a_1) - (a_2 - b_2)$. As in the previous cases we can obtain $b_1 - a_1 > h_0$.

Let $C_{n_1} = \bigcup_{\substack{j\in b_{n_1} \\ j\leq l_1}} \{j\}$. Then we have $\mu_{i_1}(C_{n_1}) - \mu_{j_1}(C_{n_1}) > h_0$.

Thus, inductively, we can obtain a sequence (C_{n_r}) of mutually disjoint elements in $\phi_0(\mathbb{N})$ and two sequences of natural numbers $(i_r)_r$ and $(j_r)_r$ with $i_1 < j_1 < \ldots < i_r < j_r < \ldots$ such that

$$\mu_{i_r}(C_{n_r}) - \mu_{j_r}(C_{n_r}) > h_0$$

for each $r \in \mathbb{N}$. This contradicts the first conclusion and so we have complete the proof of this theorem. □

ACKNOWLEDGMENTS

This project (10471124) is supported by NSFC and is supported by Zhejiang Provincial Natural Science Foundation of China (M103057) and sponsored by SRF for ROCS, SEM.

REFERENCES

Aizpuru, A. and Gutierrez-Davila, A. (2004a). Unconditionally Cauchy series and uniform convergence on Matrices. *Chinese Annals of Mathematics Series B* **25**, 335–346.

Aizpuru, A. and Gutierrez-Davila, A. (2004b). Interchange of series and applications. *Bulletin of the Belgian Mathematical Society* **11**, 409–430.

Aizpuru, A., Tamayo-Rivera, M., and Wu, J. D. (2005). Matrix convergence theorems in Quantum logics (Preprint).

Birkhoff, G. (1948). *Lattice Theory*, AMS Colloquium 25, New York.

Foulis, D. J. and Bennett, M. K. (1994). Effect algebras and unsharp quantum logics. *Foundations of Physics* **24**, 1331–1352.

Godowski, R. (1981). Varieties of orthomodular lattices with a strongly full set of states. *Demonstratio Mathematics* **14**, 725–733.

Gudder, S. (1979). *Stochastic Methods in Quantum Mechanics*: *North–Holland Series in Probability and Applied Mathematics*, North-Holland, New York.

Wu, J. D., Lu, S. J., and Kim, D. H. (2003). Antosik-Mikusinski matrix convergence theorem in quantum logics. *International Journal of Theoretical Physics* **42**, 1905–1911.

Wu, J. D, Tang, Z. F., and Cho, M. (2005). Two variables operation continuity of effect algebras. *International Journal of Theoretical Physics* **44**, 581–586.

Wu, J. D. and Ma, Z. H. (2003). The Brooks–Jewett therorem on effect algebras with the sequential completeness property. *Czechoslovak Journal of Physics* **53**, 379–383.

Wu, J. D., Lu, S. J., and Cho, M. H. (2003). Quantum-logics-valued measure convergence theorem. *International Journal of Theoretical Physics* **42**, 2603–2608.

Mazairo, F. G. (2001). Convergence theorems for topological group valued measures on effect algebras. *Bulletin of Australian Mathematics Society* **64**, 213–231.

Riecanova, Z. (2000). On order topological continuity of effect algebra operations, Contributions To General Algebra **12** (Verlag-Johannes Heyn Klagenfurt), 349–354.